Planetary g(t) for which resistive atmospheric falling is rising

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Abstract

A Darboux-transformed surface gravitational acceleration of the constant gravitational acceleration for a body endowed with an atmospheric layer is shown to turn the atmospheric free fall with quadratic resistance in the opposite motion, i.e., a free rising. Although the atmosphere of such a body may look completely normal, it is the time dependence of its gravitational field that produces this type of motion. This result is a consequence of general, one-parameter-dependent Darboux transformations in mathematical physics.

I. INTRODUCTION

The following time-dependent gravitational acceleration

$$g(t;\lambda) = g\left[1 - 2\frac{d^2}{dt^2}\ln(I_{01}(t) + \lambda)\right] , \tag{1}$$

where $I_{01}(t) = \int_0^t \cosh^2 x dx$ and λ is a parameter of the gravitational force, turns falling through a quadratic resistive - earth-like - atmosphere into just the opposite motion, i.e., a rising that for $t \to \infty$ is of constant velocity. Eq. (1) is in fact the one-parameter Darboux-transformed acceleration of the constant acceleration g at the surface of a usual planet such as Earth. The result is a consequence of a mathematical scheme that has been called strictly isospectral Darboux technique (SIDT) and has been applied extensively by one of the authors to various fields of physics. [1] Briefly, SIDT is a three step procedure that for the resistive atmospheric free fall means the following.

(i) One starts with the equation

$$\frac{dv}{dt} = g - \epsilon v^2 , \tag{2}$$

representing the normal resistive falling motion through an atmosphere of constant friction coefficient per unit of mass ϵ.

(ii) One shifts to a motion of the form

$$\frac{-dv}{dt} = g_2(t) - \epsilon v^2 \ , \tag{3}$$

where

$$g_2(t) = g[-1 + 2\tanh^2(\sqrt{\epsilon g}t)] \ . \tag{4}$$

One interpretation of Eq. (3) is that the motion is upwards through a fluid medium producing a driving force proportional to the square of the velocity of the particle (one may call it *antifriction*) plus another force $g_2(t)$ of driving character for $t < T = \frac{1}{\sqrt{\epsilon g}}\mathrm{Arctanh}\frac{1}{\sqrt{2}}$ and turning friction-like afterwards.

The first two steps are connected to each other through the particular solution of the initial normal motion as given by Eq. (7) below.

(iii) The third step is the return back to an equation similar to the initial one, namely

$$\frac{dv}{dt} = g(t;\lambda) - \epsilon v^2 \ , \tag{5}$$

where the time-dependent forces $g(t;\lambda)$ are given by Eq. (1). The second and the third steps are also connected to each other through the general solution of the second problem (see Eq. (16) below). Thus, although the motion given by Eq. (5) looks normal, its solution $v(t)$ corresponds to a rising on the vertical rather than a fall. Exactly as in the falling case, the rising has a short accelerating stage followed by a uniform rising with constant limiting velocity. Indeed, for sufficiently large times, the driving force is at its maximum value $-\epsilon V^2$, whereas $g_2(t \to \infty) = +g$ is a constant friction force and one gets the value of V from the balance of the two forces.

Here, we first review the usual resistive atmospheric falling according to a 40-year old discussion of Davis. [2] Next, we briefly describe the motion in a Darboux-transformed gravitational field. Some concluding remarks end up the work.

II. DAVIS' RESISTIVE FREE FALL

The free fall in quadratic resistive media has been studied by Davies in his book as a simple application of the Riccati equation for the falling velocity. Davis wrote the "readily found" solution of Eq. (2) in the following form

$$v(t) = V\left(\frac{u_0 + \tanh\sqrt{\epsilon g}t}{1 + u_0\tanh\sqrt{\alpha g}t}\right) \ , \tag{6}$$

where $u_0 = v_0/V$ and $V = \sqrt{g/\epsilon}$ is the so-called limiting (terminal) velocity. Notice that Eq. (6) can also be written as

$$v(t) = V\tanh(\sqrt{\epsilon g}t + \beta) \ , \tag{7}$$

where the arbitrary phase β is fixed through the initial condition, $\beta = \mathrm{Arctanh}u_0$.

Davis obtained the falling distance $s_D(t)$ by integrating the velocity:

$$s_D(t) = \int_0^t v(t)dt = \frac{V^2}{g}\ln\left(\cosh\frac{gt}{V} + \frac{v_0}{V}\sinh\frac{gt}{V}\right) \ . \tag{8}$$

The comparison with a set of 25 experimental data of parachute falls in the atmosphere of the earth was presented by Davis for the particular case $u_0 = 0$, i.e. $v_p = V\tanh\sqrt{\epsilon g}t$ and

$$s_D = \frac{V^2}{g}\ln\left(\cosh\frac{gt}{V}\right) \ \approx Vt - \frac{V^2}{g}\ln 2 \ , \tag{9}$$

where the last approximation is for large times. The motion s_D is asymptotically a uniform falling with the limiting velocity V.

III. ONE-PARAMETER DARBOUX TRANSFORMED FORCES AND RESULTING MOTION

We now provide details of the Darboux constructions. We recall that nonrelativistic supersymmetric quantum mechanics is a simple application of Darboux transformations, with a huge publication output during the last two decades. One usually starts with a Riccati equation, such as Eq. (2), that we write in dimensionless form

$$\mathcal{R}_1 : \quad \frac{du}{d\tau} + u^2 = 1 \ , \tag{10}$$

having $u_1 = \tanh(\tau + \gamma)$ as solution. Next, it is easy to see that the right hand side corresponds up to a sign to a constant potential function for the Schrödinger equation at zero energy

$$\left(\frac{d}{d\tau} + u_1\right)\left(\frac{d}{d\tau} - u_1\right)w_1 = \frac{d^2w_1}{d\tau^2} - w_1 = 0 \ . \tag{11}$$

The particular solution $w_1 = \cosh(\tau + \gamma)$ is usually called a zero mode. It is connected to the Riccati solution through $u_1 = \frac{1}{w_1}dw_1/d\tau$. Next, changing the sign of the first derivative in $\mathcal{R}_1$ one calculates the outcome $\tilde{g}_f(\tau)$ using the same Riccati solution u_1, i.e., $-\frac{du_1}{d\tau} + u_1^2 = 2\tanh^2(\tau + \gamma) - 1 := \tilde{g}_f(\tau)$. Thus, one considers a second Riccati equation

$$\mathcal{R}_2 : \quad -\frac{du}{d\tau} + u^2 = 2\tanh^2(\tau + \gamma) - 1 \ , \tag{12}$$

of which we already know that it has u_1 as solution. The corresponding zero mode $w_2 = 1/\cosh(\tau + \gamma)$ fulfills

$$\left(\frac{d}{d\tau} - u_1\right)\left(\frac{d}{d\tau} + u_1\right)w_2 = \frac{d^2w_2}{d\tau^2} - \tilde{g}_f(\tau)w_2 = 0 \ . \tag{13}$$

We are now interested in the general solution u_{g2} of $\mathcal{R}_2$. To find it, one employs the Bernoulli ansatz $u_{g2}(\tau) = u_1(\tau) - \frac{1}{f(\tau)}$, where $f(\tau)$ is an unknown function. One obtains for the function $f(\tau)$ the following Bernoulli equation

$$\frac{df(\tau)}{d\tau} + 2f(\tau)\, u_1(\tau) = 1 \ , \tag{14}$$

that has the solution

$$f(\tau) = \frac{\mathcal{I}_{0b}(\tau) + \lambda}{w_1^2(\tau)} \ , \tag{15}$$

where $\mathcal{I}_{01}(\tau) = \int_0^\tau w_1^2(y)\, dy$, and we consider λ as a positive integration constant that is employed as a free parameter of the force field.

Thus, the general Riccati solution of $\mathcal{R}_2$ is a two-parameter function $u_{g2}(\tau; \gamma, \lambda)$ of the following form

$$u_{g2}(\tau; \beta; \lambda) = \frac{d}{d\tau}\Big[\ln\left(\frac{w_1(\tau)}{\mathcal{I}_{01}(\tau) + \lambda}\right)\Big] = \frac{d}{d\tau}\Big[\ln\left(\frac{\cosh(\tau + \gamma)}{\frac{1}{4}\sinh 2(\tau + \gamma) + \frac{1}{2}(\tau + \gamma) + \lambda}\right)\Big] \ , \tag{16}$$

where in the last λ we absorbed the lower limit of the integral. The range of the λ parameter is conditioned by $\mathcal{I}_0(\tau) + \lambda \neq 0$ in order to avoid singularities. According to SIDT one can use u_{g2} to calculate a family of force functions $\tilde{g}_\lambda := \frac{du_{g2}}{d\tau} + u_{g2}^2$. Thus

$$\tilde{g}_\lambda = 1 - 2\frac{d^2}{d\tau^2}\ln\left(\mathcal{I}_{01}(\tau) + \lambda\right) \ . \tag{17}$$

This expression coincides with Eq. (1) when passed to dimensional quantities. Thus, there is a third Riccati equation, namely

$$\mathcal{R}_3 : \quad \frac{du}{d\tau} + u^2 = \tilde{g}_\lambda \ , \tag{18}$$

which is similar in form to $\mathcal{R}_1$ but possessing the solution u_{g2}. The corresponding linear equation is

$$\left(\frac{d}{d\tau} + u_{g2}\right)\left(\frac{d}{d\tau} - u_{g2}\right) w_3 = \frac{d^2 w_3}{d\tau^2} - \tilde{g}_\lambda w_3 = 0 \ , \tag{19}$$

where

$$w_3 = \left(\frac{w_1(\tau)}{\mathcal{I}_{01}(\tau) + \lambda}\right) \ . \tag{20}$$

In the limit $\lambda \to \infty$, Eq. (18) goes into Eq. (10) because $u_{g2} \to u_1$ and $\tilde{g}_\lambda \to 1$. Thus, one can say that Eq. (18) is the one-parameter SIDT generalization of the Riccati equation (10).

The problem of imposing initial conditions can be solved similarly to Davis. Since the solution u_{g2} has two parameters, γ and λ, we shall fix the phase parameter through the initial condition $u_{g2}(0) = u_0$ and the other parameter. This leads to the following cubic algebraic equation

$$\tanh^3\gamma - u_0\tanh^2\gamma - \tanh\gamma = -(u_0 + \frac{1}{\lambda}) \ . \tag{21}$$

The three solutions of this equation can be obtained by Cardano's formulas and any of them can be used to fix γ. The real one is the most simple

$$\gamma_1 = \mathrm{Arctanh}\frac{1}{3}\left(u_0 - \frac{E^{1/3} - E^{-1/3}}{\lambda}\right) \ , \tag{22}$$

where $E = \left(\frac{B+\sqrt{4A^3+B^2}}{2}\right)^{1/3}$, $A = -3\lambda^3 - \lambda^2 u_0$, and $B = 27\lambda^2 + 27\lambda^3 u_0 - 9\lambda^4 u_0 - 2\lambda^3 u_0^3$. The other two solutions are complex and being more complicated will not be reproduced here.

Let us calculate the distance in the motion $\mathcal{R}_3$. It is given by

$$s_3(\tau) = \int_0^\tau u_{g2} = \ln\left(\frac{w_1(\tau)}{\mathcal{I}_{01}(\tau) + \lambda}\right) \approx \ln\left(4e^\tau e^{-2\tau}\right) \to -Vt + \frac{2V^2}{g}\ln 2 \tag{23}$$

that shows that in the large-time asymptotics the motion is a vertical rising of constant velocity V.

IV. CONCLUSION

One may say that if there exist planetary-like astrophysical bodies with gravitational fields of the type given by Eq. (1), then an anti-parachute effect might occur in their atmospheric layer. It is hard to believe that nature can produce such fields. For example, in the case of the common SIDT application to supersymmetric quantum mechanics, the scheme is used merely to extend the class of exactly solvable potentials and in general is considered to be a curiosity. However, we point out the debate on the modifications of the gravitational fields in superconductors as a consequence of macroscopic quantum effects with applications to neutron stars. [3]

Finally, we would like to make the following comment. SIDT introduces singularities in all the transformed quantities. The parameter λ gives the measure of the time interval where the singularity is located in the past ($t < 0$). A small λ means a singularity close to the moment of parachute jumps and the time dependence is significant. On the other hand, a large λ leads to very small variation in time of the planetary gravitational field that can be thought of as a cosmological evolution implying a singularity very far in the past.

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[2] H. Davis, *Introduction to nonlinear differential and integral equations*, (Dover Publications, Inc., New York, 1962), pp. 60-62.

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Appendix

In general, for a Riccati equation of the type

$$\frac{dy}{dx} = \alpha y^2 + \beta \ , \qquad \alpha\, , \beta = \text{const.} \tag{A1}$$

the solutions can be written as follows

(i) for $\frac{\alpha}{\beta} > 0$

$$y = \sqrt{\frac{\beta}{\alpha}} \tan\left(\sqrt{\frac{\alpha}{\beta}}\beta x + \phi_1\right), \qquad \text{or} \qquad y = \sqrt{\frac{\beta}{\alpha}} \cot\left(-\sqrt{\frac{\alpha}{\beta}}\beta x + \phi_2\right) \tag{A2}$$

(ii) for $\frac{\alpha}{\beta} < 0$

$$y = \sqrt{-\frac{\beta}{\alpha}} \tanh\left(\sqrt{-\frac{\alpha}{\beta}}\beta x + \varphi_1\right), \quad \text{or} \quad y = \sqrt{-\frac{\beta}{\alpha}} \coth\left(\sqrt{-\frac{\alpha}{\beta}}\beta x + \varphi_2\right) \ . \tag{A3}$$

Comparing the Newtonian falling case with Eq. (A1) one gets

$$\alpha = -\epsilon, \qquad \beta = g \qquad \Longrightarrow \quad \frac{\alpha}{\beta} = \frac{-\epsilon}{g} < 0, \tag{A4}$$

and therefore the solution is given by

$$v(t) = \sqrt{\frac{g}{\epsilon}} \tanh\left(\sqrt{\epsilon g}t + \varphi_1\right) \ , \tag{A5}$$

which is Davis' result. The falling velocity $v(t)$ presents a substantial time dependence - acceleration stage - only for a short period and rapidly turns into the limiting constant velocity $V = \sqrt{\frac{g}{\epsilon}}$.

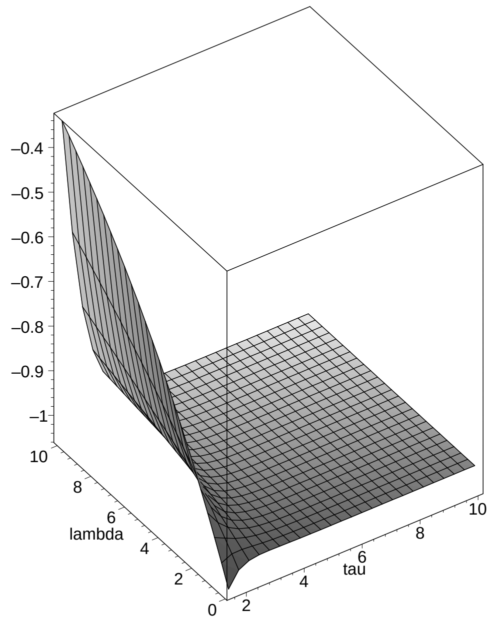

Fig. 1: The velocity u_{g2} for $\gamma_1 = 1$.

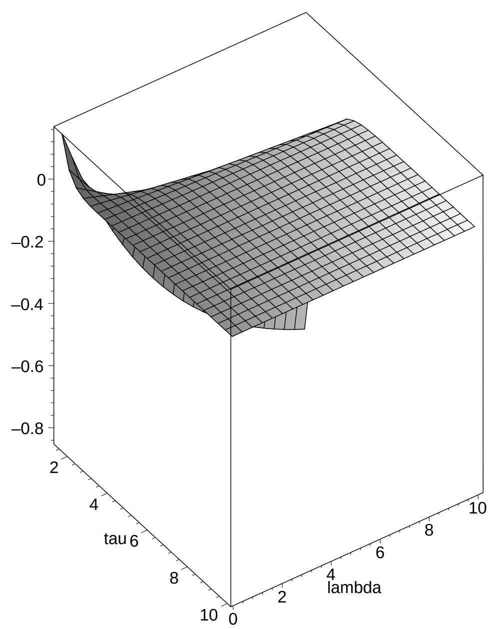

Fig. 2: The derivative of u_{g2} (the acceleration) for the same γ_1.

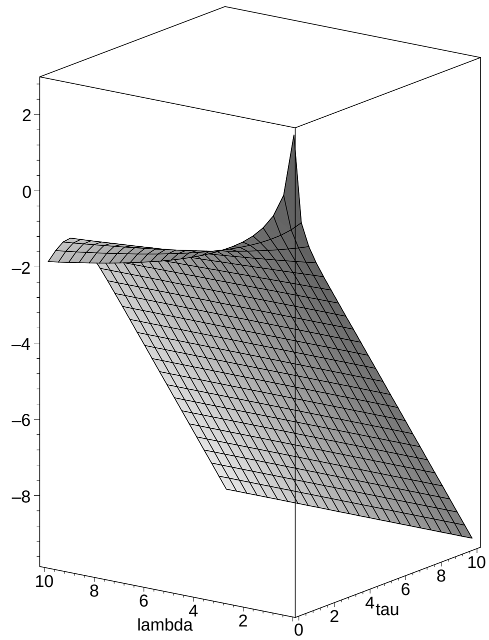

Fig. 3: The rising distance s_3 for the same γ_1.

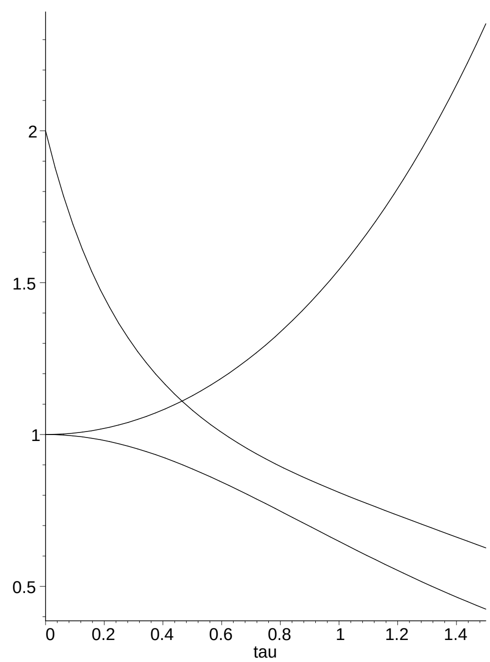

Fig. 4 The zero modes w_1, w_2, and $w_3(\lambda = 0.5)$.

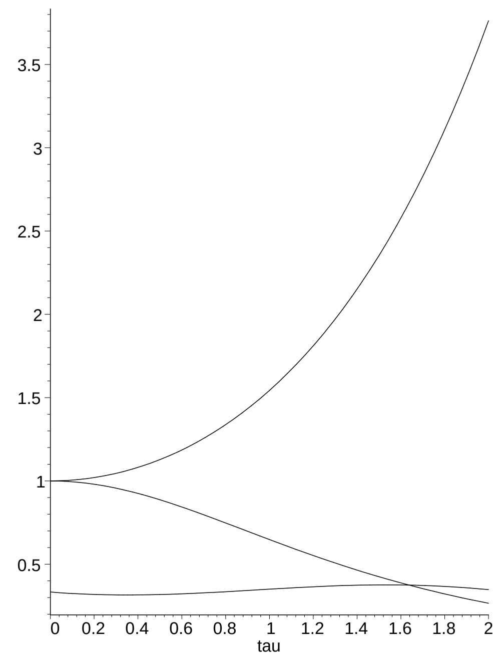

Fig. 5   The zero modes w_1, w_2, and $w_3(\lambda = 3)$.

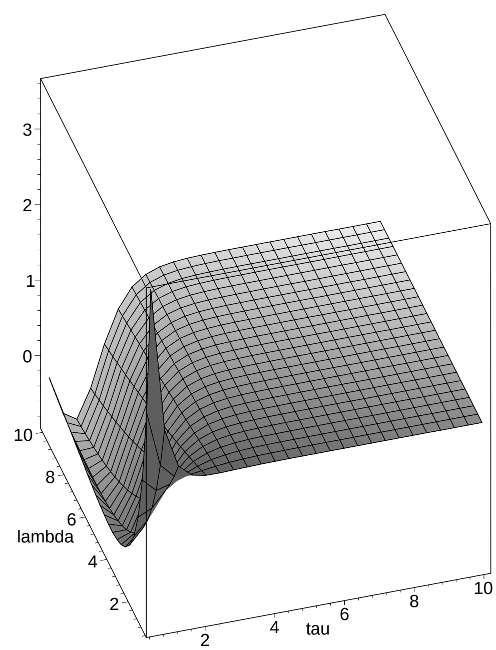

Fig. 6   The normalized force $\tilde{g}_\lambda$ for $\gamma_1 = 1$.